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STEADY AND TRANSIENT HEAT TRANSFER BY RADIATION AND CONDUCTION IN A MEDIUM BOUNDED BY TWO COAXIAL CYLINDRICAL SURFACES

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Abstract—The heat transfer in a conducting, emitting and absorbing medium bounded by two infinite coaxial cylindrical surfaces is analyzed. Both transient and steady states are considered. According to the quasi-steady simplification and Eddington's first approximation for radiative transfer, the problem is formulated in two differential equations: one for the radiation potential and the other for the temperature. They are then transformed into integral equations which are solved by the method of successive approximations. Some insight into the effect of radiation on the heat flow is afforded by the calculation of radiation potential. For highly emissive surfaces, it is found that the interaction of radiation with conduction has a negligible effect on the total heat flux for both steady and transient states. Hence simple formulas are obtained for the calculation of heat fluxes.

NOMENCLATURE

c_p,	specific heat;
G_ϕ,	Green's function of radiation potential;
G_T,	Green's function of temperature;
H_0,	defined by equation (13);
h or h_s,	$3\,\varepsilon_s/[2(2-\varepsilon_s)]$;
I_n,	modified Bessel function of first kind and order n;
I^*,	intensity of radiation;
J_n,	Bessel function of first kind and order n;
K_n,	modified Bessel function of second kind and order n;
k,	thermal conductivity;
N,	$k\kappa/(4\sigma T_1^{*3})$;
n,	inward drawn normal to surface;
$\vec{q}^*$,	heat flux;
$\vec{q}$,	$\vec{q}^*/(\sigma T_1^{*4})$;
r^*,	radial coordinate;
r,	κr^*;
T^*,	temperature;
T,	T^*/T_1^*;
t^*,	time;
t,	$4\kappa\sigma T_1^{*3}t^*/(\rho c_p)$;
Y_n,	Bessel function of second kind and order n;
ε,	emissivity;
κ,	absorption coefficient;
λ_n,	eigenvalues of equation (14);
ϕ,	$\pi I^*/(\sigma T_1^{*4})$, radiation potential;
ϕ_1,	defined by equation (6);
ρ,	density of medium;
σ,	Stefan–Boltzmann constant;
τ,	$\kappa(r_2 - r_1)$, optical thickness of medium.

Subscripts

b,	black radiation;
c,	pertaining to heat conduction;
r,	pertaining to radiative heat transfer;
s,	surface;
1, 2,	surfaces at r_1 and r_2.

Superscripts

$*$,	dimensional quantity;
$'$,	pertaining to source in Green's functions;

$\bar{\ }$, average or Laplace transformed quantity.

INTRODUCTION

IN RECENT years, many papers have been published on the heat transfer in steady state by combined radiation and conduction through a plane medium. There have been few investigations for other systems, particularly in transient state. Howell [1] studied the steady problem for the energy transfer in media bounded by two parallel plane surfaces and also two coaxial cylindrical surfaces through the method of exchange factor approximation. Greif and Clapper [2] calculated the heat transfer in an annular medium by the superposition of pure conduction and pure radiation and found that the results are in good agreement with those reported in [1]. Viskanta and Merriam [3] investigated the steady problem for a hollow spherical medium through the solution of the rigorously formulated integral and integro-differential equations. Lick [4] considered the transient energy transfer in a semi-infinite medium bounded by a non-emitting and non-reflecting surface by linearizing the re-emission term. Nemchinov [5] also studied the linearized problem but was interested only in the propagation of thermal waves. Viskanta and Lall [6] studied the transient problem of heat transfer in a spherical medium at first from the exactly formulated integral and integro-differential equations but diverted later to approximate calculations. Most recently, the steady and transient problems of heat transfer in a plane medium have been analyzed by Chang and Kang [7] according to a potential formulation.

This paper concerns the steady and transient heat transfer in a conducting, absorbing and emitting medium bounded by two infinite coaxial cylindrical surfaces. The fundamental differential equations are formulated by employing the first approximation of Eddington for radiative transfer. These differential equations are then transformed, by the use of Green's functions, into integral equations which are solved by the method of successive approximations.

One purpose of studying the steady problem is to indicate once more the usefulness of the differential formulation and the method of solution which has proved to yield accurate results for the temperature field and heat flux in plane layers. Another purpose is to suggest some approximate methods by which the heat flux can be calculated and how the non-linear problem may be linearized.

The transient study deals with the response of the temperature and heat flow to a sudden change of the boundary conditions and indicates that this difficult problem can be treated with facility by the same method as for the steady state.

The formulation of fundamental equations has been reported by several authors [8–10]. However, the equations, particularly the boundary conditions, can be obtained in a somewhat different manner. Thus, for quick reference, they are derived in the Appendix.

STATEMENT OF THE PROBLEM AND BASIC ASSUMPTIONS

The problem to be considered is illustrated in Fig. 1. A conducting, emitting and absorbing medium is bounded by two infinite coaxial cylindrical surfaces of radii r_1^* and r_2^*. The medium is initially at absolute zero temperature. The surfaces are suddenly brought to and kept at constant and uniform temperatures $T_1^*(r_1^*)$ and $T_2^*(r_2^*)$. As time approaches infinity, the steady state is reached.

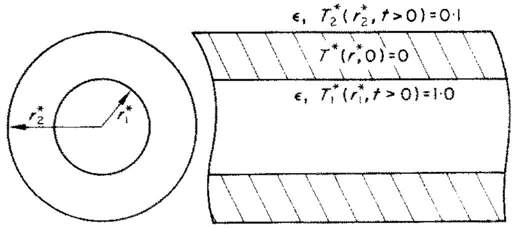

FIG. 1. Geometry for medium between infinite concentric cylinders.

We assume that radiation is locally in thermodynamic equilibrium and quasi-steady at any instant of time, scattering is negligible, and the medium is grey with a unit refractive index. We also assume that the physical and thermal properties of the medium are constant and the surfaces are grey and emit and reflect radiation diffusely.

Transient state

In the transient state the governing equations of the radiation potential and the temperature are given by equations (A.4) and (A.12) in the Appendix. By introducing the dimensionless quantities as defined in the Nomenclature and specializing to the present problem they become

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - 3\phi = -3T^4 \qquad (1)$$

$$\frac{\partial T}{\partial t} - N\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) = \phi - T^4. \qquad (2)$$

If emissivities of the two surfaces are assumed as constant and the same, the boundary conditions on $\phi(r, t)$ are obtained from (A.7) as

$$\begin{aligned} \frac{\partial\phi}{\partial r} &= h(\phi - T_1^4) \qquad \text{at} \quad r = r_1 \\ \frac{\partial\phi}{\partial r} &= -h(\phi - T_2^4) \qquad \text{at} \quad r = r_2. \end{aligned} \qquad (3)$$

The boundary conditions on $T(r, t)$, as prescribed earlier, are

$$\begin{aligned} T(r_1, t) &= T_1 \qquad T(r_2, t) = T_2. \\ T(r, 0) &= 0 \end{aligned} \qquad (4)$$

Note that the radiation potential ϕ changes with time only because of the presence of the source term, $3T^4(r, t)$.

We now wish to solve (1) and (2) by the method of successive approximations. For this purpose, it is convenient to transform them into integral equations by considering quantities on the right-hand sides as known functions so that the method of Green's function can be used.

If $G_\phi(r|r')$ is the Green's function associated with ϕ, the formal solution for ϕ satisfying (1) and (3) can then be written as

$$\phi(r, t) = \phi_1(r) + 6\pi \int_{r_1}^{r_2} G_\phi(r|r')\, T^4(r', t)\, r'\, dr'. \qquad (5)$$

In (5) $\phi_1(r)$ is the homogeneous solution of (1) satisfying (3) and is obtained as

$$\phi_1(r) = A_0 I_0(\sqrt{3}r) + B_0 K_0(\sqrt{3}r) \qquad (6)$$

where I_0 and K_0 are modified Bessel functions of the first and second kinds and

$$\begin{aligned} A_0 &= -\frac{h}{C_0}[T_1^4 F_1(\sqrt{3}r_2) + T_2^4 F_3(\sqrt{3}r_1)] \\ B_0 &= -\frac{h}{C_0}[T_2^4 F_2(\sqrt{3}r_1) + T_1^4 F_4(\sqrt{3}r_2)], \end{aligned} \qquad (7)$$

with

$$\begin{aligned} F_1(\sqrt{3}r_2) &= \sqrt{3}K_1(\sqrt{3}r_2) - hK_0(\sqrt{3}r_2) \\ F_2(\sqrt{3}r_1) &= \sqrt{3}I_1(\sqrt{3}r_1) - hI_0(\sqrt{3}r_1) \\ F_3(\sqrt{3}r_1) &= \sqrt{3}K_1(\sqrt{3}r_1) + hK_0(\sqrt{3}r_1) \\ F_4(\sqrt{3}r_2) &= \sqrt{3}I_1(\sqrt{3}r_2) + hI_0(\sqrt{3}r_2) \\ C_0 &= F_1(\sqrt{3}r_2)\, F_2(\sqrt{3}r_1) - F_3(\sqrt{3}r_1)\, F_4(\sqrt{3}r_2). \end{aligned} \qquad (8)$$

The Green's function, $G_\phi(r|r')$ in (5) can be obtained from its properties by writing

$$G_\phi(r|r') = \frac{1}{2\pi} I_0(\sqrt{3}r')\, K_0(\sqrt{3}r) + C_1(r')\, I_0(\sqrt{3}r) + C_2(r')\, K_0(\sqrt{3}r) \qquad \text{for} \quad r > r'$$

$$G_\phi(r|r') = \frac{1}{2\pi} I_0(\sqrt{3}r)\, K_0(\sqrt{3}r') + C_1(r')\, I_0(\sqrt{3}r) + C_2(r')\, K_0(\sqrt{3}r) \qquad \text{for} \quad r < r',$$

where the first term is the cylindrical surface source and sum of the second and third terms is the solution satisfying the homogeneous part of (1). Evaluating $C_1(r')$ and $C_2(r')$ from the following conditions

$$\frac{\mathrm{d}\,G_\phi}{\mathrm{d}\,r} = h\,G_\phi \qquad \text{at} \quad r = r_1$$

$$\frac{\mathrm{d}\,G_\phi}{\mathrm{d}\,r} = -h\,G_\phi \qquad \text{at} \quad r = r_2$$

we obtain, after a long procedure of deduction, for $r > r'$

$$G_\phi(r|r') = [AK_0(\sqrt{3}r') + BI_0(\sqrt{3}r')]\,I_0(\sqrt{3}r) + [CI_0(\sqrt{3}r') + DK_0(\sqrt{3}r')]\,K_0(\sqrt{3}r) \qquad (9)$$

For $r < r'$, r and r' are interchanged in (9). Coefficients in (9) are given as follows:

$$\begin{aligned} A &= F_1(\sqrt{3}r_2)\,F_2(\sqrt{3}r_1)/E \\ B &= F_3(\sqrt{3}r_1)\,F_1(\sqrt{3}r_2)/E \\ C &= F_3(\sqrt{3}r_1)\,F_4(\sqrt{3}r_2)/E \\ D &= F_4(\sqrt{3}r_2)\,F_2(\sqrt{3}r_1)/E \end{aligned} \qquad (10)$$

where

$$E = 2\pi[F_3(\sqrt{3}r_1)\,F_4(\sqrt{3}r_2) - F_2(\sqrt{3}r_1)\,F_1(\sqrt{3}r_2)]$$

The formal solution of (2) for T can be found in the same way. If $G_T(r, t|r', t')$ is the Green's function associated with $T(r, t)$, the formal solution for T is then

$$T(r, t) = T_c(r, t) + 2\pi \int_0^t \mathrm{d}t' \int_{r_1}^{r_2} G_T(r, t|r', t')\,[\phi(r', t') - T^4(r', t')]\,r'\,\mathrm{d}r' \qquad (11)$$

where $T_c(r, t)$ is the solution satisfying the homogeneous part of (2) and the boundary conditions (4), i.e. the solution of pure conduction and is well-known [11],

$$T_c(r, t) = -\pi \sum_{n=1}^{\infty} \frac{[T_2 J_0(\lambda_n r_1) - T_1 J_0(\lambda_n r_2)]\,J_0(\lambda_n r_2)\,H_0(\lambda_n r)}{J_0^2(\lambda_n r_1) - J_0^2(\lambda_n r_2)}\,\mathrm{e}^{-N\lambda_n^2 t} + \frac{T_1 \ln(r_2/r) - T_2 \ln(r/r_1)}{\ln(r_2/r_1)}, \qquad (12)$$

where

$$H_0(\lambda_n z) = Y_0(\lambda_n r_1)\,J_0(\lambda_n z) - J_0(\lambda_n r_1)\,Y_0(\lambda_n z) \qquad (13)$$

and λ_n are the roots of

$$J_0(\lambda r_1)\,Y_0(\lambda r_2) - Y_0(\lambda r_1)\,J_0(\lambda r_2) = 0. \qquad (14)$$

The Green's function, $G_T(r, t|r', t')$ in (11) can be found by Eigen-function expansion [12], or Laplace transformation. The Laplace transform of $G_T(r, t|r', t')$, i.e. $\overline{G}_T(r|r', p)$, can be readily obtained from (9) by setting $\sqrt{3} = \sqrt{p}$ and $h \to \infty$. By inversion theorem we obtain,

$$G_T(r, t|r', t') = \frac{\pi}{4} \sum_{n=1}^{\infty} \lambda_n^2 \frac{J_0^2(\lambda_n r_2)\,H_0(\lambda_n r)\,H_0(\lambda_n r')}{J_0^2(\lambda_n r_1) - J_0^2(\lambda_n r_2)} \times \mathrm{e}^{-N\lambda_n^2(t-t')}, \qquad (15)$$

which has been given in [11].

Equations (5) and (11) are two integral equations which are most suitable for iterative solution. Once the temperature and radiation potential are known, the total heat flux can be calculated by using (A.13)

$$q = -4N\frac{\partial T}{\partial r} - \frac{4}{3}\frac{\partial \phi}{\partial r} = q_c(r, t) + q_r(r, t). \qquad (16)$$

Steady state

For steady state, the formal solution for ϕ is obtained from (5) by dropping the parameter t, while $\phi_1(r)$ and $G_\phi(r|r')$ remain the same as (6) and (9). The formal solution for $T(r)$ as well as its Green's function can be obtained by letting $t \to \infty$ in (11), (12) and (15), but the reduction of resulting infinite series to simple expressions of closed form is tedious. Furthermore, it is more convenient to use equation (A.10) rather than (A.12) in the Appendix for steady state. By non-dimensionalizing and specializing (A.10) to the present problem, we obtain

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)-\frac{1}{3Nr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)=0. \quad (17)$$

Integrating this twice and using the boundary conditions

$$\begin{aligned} T &= T_1, \quad \phi = \phi(r_1) \quad \text{at} \quad r = r_1 \\ T &= T_2, \quad \phi = \phi(r_2) \quad \text{at} \quad r = r_2, \end{aligned} \quad (18)$$

we obtain

$$\begin{aligned} T = T_1 &- \frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_1 - T_2) \\ &- \frac{\ln(r/r_1)}{3N\ln/r_2/r_1)}[\phi(r_1) - \phi/r_2)] \\ &+ \frac{1}{3N}[\phi(r_1) - \phi(r)]. \end{aligned} \quad (19)$$

Substituting $\phi(r)$, given in (5), into (19) yields an integral equation which can be again solved by the method of successive approximations.

For very small values of N, the iteration on (19) will converge slowly. For this case, we can apply integration by parts to the integral in (5) so that the parameter $1/N$ in front of the integral in (19) can be removed. Thus, an alternative form of (19) which is suitable for small values of N is obtained as

$$\begin{aligned} T^4 + 3NT &= 3NT_1 - 3N(T_1 - T_2) \\ &\times \frac{\ln(r/r_1)}{\ln(r/r_1)} - [\phi(r_1) - \phi(r_2)]\frac{\ln(r/r_1)}{\ln(r_2/r_1)} \\ &+ \phi(r_1) - \Psi(r), \end{aligned} \quad (20)$$

where

$$\begin{aligned} \Psi(r) &= \phi_1(r) + \frac{6\pi}{\sqrt{3}}[r_1T_1^4L(r,r_1) - r_2T_2^4M(r,r_2)] \\ &- \frac{6\pi}{\sqrt{3}}\int_{r_1}^{r} L(r,r')\,r'\frac{\mathrm{d}T^4}{\mathrm{d}r'}\mathrm{d}r' - \frac{6\pi}{\sqrt{3}} \\ &\times \int_{r}^{r_2} M(r,r')\,r'\frac{\mathrm{d}T^4}{\mathrm{d}r'}\mathrm{d}r' \end{aligned} \quad (21)$$

In (21) the functions $L(r, z)$ and $M(r, z)$ are given by

$$\begin{aligned} L(r,z) &= [BI_0(\sqrt{3}r) + CK_0(\sqrt{3}r)]\,I_1(\sqrt{3}z) \\ &\quad - [AI_0(\sqrt{3}r) + DK_0(\sqrt{3}r)]\,K_1(\sqrt{3}z) \\ \mathrm{M}(r,z) &= [BI_0(\sqrt{3}r) + AK_0(\sqrt{3}r)]\,I_1(\sqrt{3}z) \\ &\quad - [CI_0(\sqrt{3}r) + DK_0(\sqrt{3}r)]\,K_1(\sqrt{3}z), \end{aligned} \quad (22)$$

where A, B, C and D are given by (10).

The total heat flux is obtained from (A.13) as

$$q = q_c + q_r = \frac{4N(T_1 - T_2)}{r\ln(r_2/r_1)} + \frac{\phi(r_1) - \phi(r_2)}{(3r/4)\ln(r_2/r_1)}. \quad (23)$$

Note from (23) that once the radiation potential is known, the heat flux can be readily calculated. Note also that if $[\phi(r_1) - \phi(r_2)]$ were independent of N there would not be interaction between radiation and conduction.

NUMERICAL SOLUTION AND RESULTS

Equations (5) and (11) for transient state and (5) and (19) for steady state were numerically solved by the method of successive approximations on a CDC-6400 computer. The integration and interpolation were made according to the spline-fit approximation [13] and the sub-routines of Katsanis [14] were applied. Other methods of approximation had been tried, but it was found that the spline-fit method is more accurate and efficient in the solution of integral equations, because the mesh size can be changed between calculation points whenever it is required. However, its use in the calculation of derivatives is restrictive as was noted by Katsanis [14]. Therefore, the calculation of the heat flux where the temperature gradient changes suddenly may not be sufficiently accurate. However, only the wall fluxes are required and hence no difficulty was encountered.

The Green's functions were first calculated. For the transient state, the explicit part of the right-hand side of (11) was taken as the first approximation of $T(r, t)$, i.e.,

$$T^{(1)}(r, t) = T_c(r, t) + 2\pi \int_0^t \mathrm{d}t' \int_{r_1}^{r_2} \phi_1(r')\, G_T(r, t \,|\, r', t')\, r'\, \mathrm{d}r' \tag{24}$$

To save the core storage in the computer, the solution of (11) for $T(r, t)$ was done by dividing t into several steps. First, $T(r, t)$ was calculated up to a small value of t, say t_1. Then $T(r, t)$ was calculated for $t_1 \leqslant t \leqslant t_2$ by using $T(r, t_1)$ as the initial temperature and adding to the right-hand side of (11) the expression

$$\frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{\lambda_n^2 J_0^2(\lambda_n r_2)}{J_0^2(\lambda_n r_1) - J_0^2(\lambda_n r_2)} \mathrm{e}^{-N\lambda_n^2 t} H_0(\lambda_n r) \times \int_{r_1}^{r_2} r' T(r', t_1)\, H_0(\lambda_n r')\, \mathrm{d}r' \tag{25}$$

The calculations for $t_n \leqslant t \leqslant t_{n+1}$ were done in the same manner.

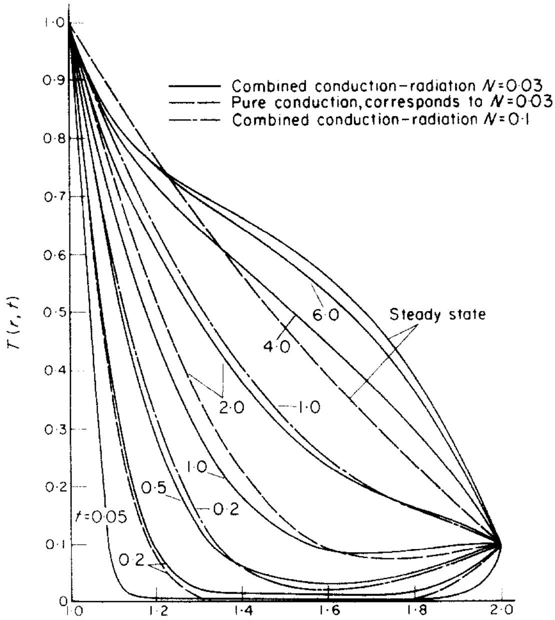

FIG. 3. Temperature in transient state for $h = 0{\cdot}5$.

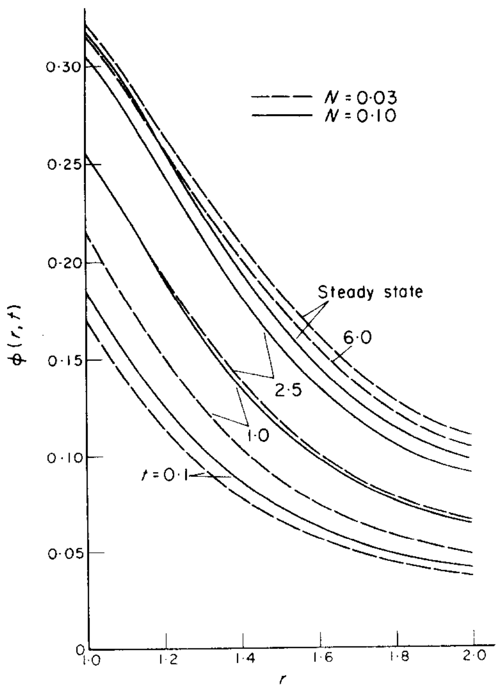

FIG. 2. Distribution of radiation potential in transient state for $T_1 = 1$, $T_2 = 0{\cdot}1$, $h = 0{\cdot}5$.

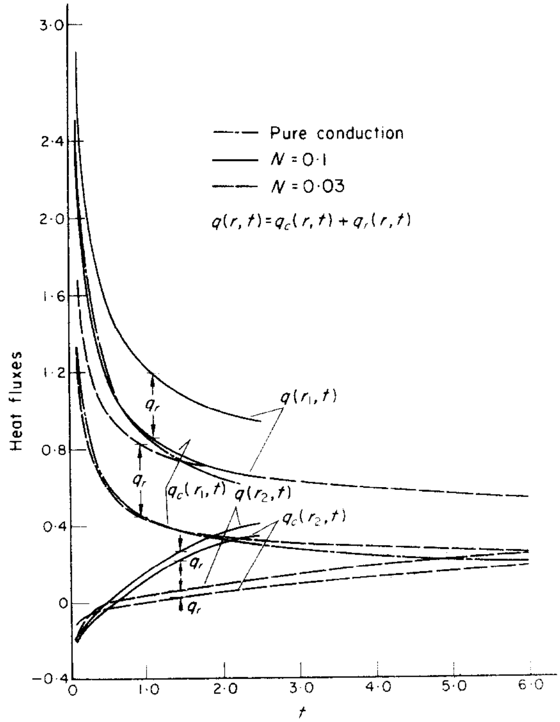

FIG. 4. Heat fluxes in transient state for $T_1 = 1$, $T_2 = 0{\cdot}1$, $h = 0{\cdot}5$.

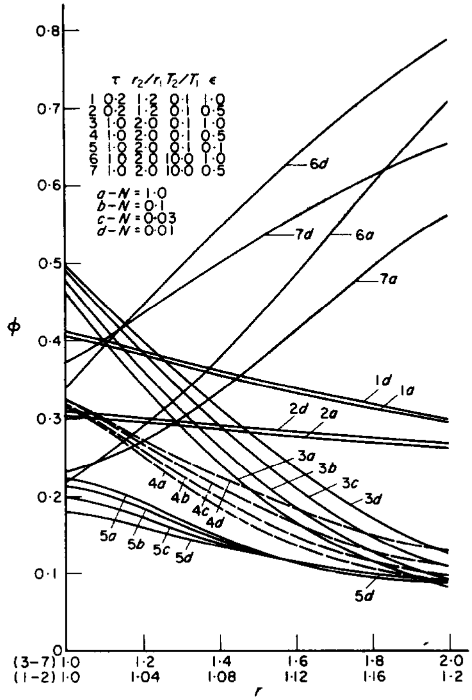

FIG. 5. Distribution of radiation potential in steady state.

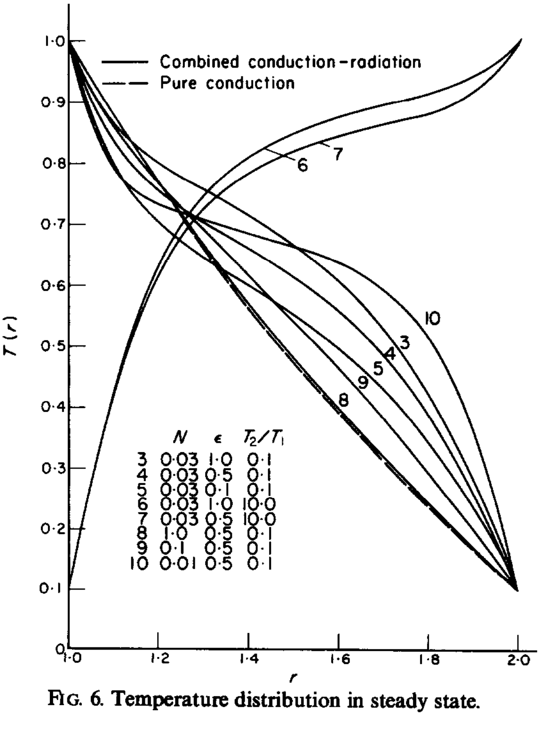

FIG. 6. Temperature distribution in steady state.

Generation of the Green's functions and other data required about 50 s of computer time. Each interation required 8 s. Maximum error was assigned at 0·1 per cent for convergence. Thirteen points along the radial coordinate were found to yield results with sufficient accuracy. Of these four were grouped in a finer mesh about the source point and could be moved as the source point changed. For small values of t two or three iterations yielded the desired accuracy.

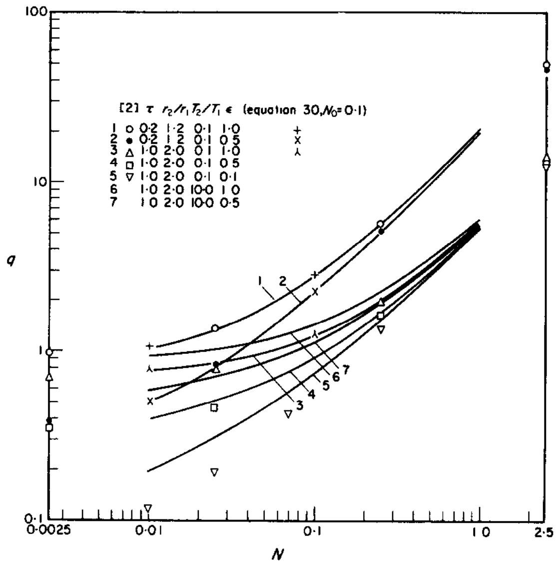

FIG. 7. Heat fluxes in steady state at inner surface.

For the steady state, the iteration on (5) for $\phi(r)$ and on (19) for $T(r)$ are much simpler than those for the transient state. Each iteration required less than 0·1 s.

Some of the calculated results of the radiation potentials for given values of the various parameters are shown in Figs. 2 and 5, of the temperature fields in Figs. 3 and 6 and of the heat fluxes in Figs. 4 and 7. Temperatures for pure conduction are also plotted in Figs. 3 and 6 for comparison.

Discussion and approximate formulae for heat flux

The potential fields of radiation, $\phi(r, t)$ and $\phi(r)$, as shown in Figs. 2 and 5, are of particular interest. They not only give a greater insight into the effect of radiation on the temperature fields but also lead to some simple, approximate formulae for the prediction of the heat flux.

Consider first the transient state. Calculations were carried out for $N = 0{\cdot}1$ and $N = 0{\cdot}03$. Figure 2 shows that the gradient of radiation potential at the hot surface (i.e. the inner surface) is always larger in magnitude than that at the cold surface. This indicates that radiant energy is stored which elevates the temperature in the medium. Thus, the smaller the value of N the faster the increase of temperature gradient at the outer surface which will, in turn, augment the heat transfer by conduction, as clearly shown in Fig. 3. Figure 2 also shows that $\partial\phi/\partial r$ decreases (in magnitude) with the increase of time at the inner surface but increases at the outer surface, i.e. the radiant energy trapped in the medium is smaller at later times. These two factors will, consequently, reduce the temperature near the inner wall below that of pure conduction at larger times. Since the inner wall temperature is maintained at a constant value, the temperature profile then exhibits an S-shape. This is particularly evident for the steady state distribution, as clearly shown in Fig. 6 for the small values of N.

Figure 4 shows that at the inner surface the total heat flux decreases with time faster at earlier times, but the radiant heat flux decreases slowly all the time. Note that the difference between the total heat flux and the heat flux for pure conduction is nearly independent of N and t for $t > 0{\cdot}5$. This indicates that as far as the total heat flux is concerned, the interaction of radiation with thermal capacity and conduction is negligible for cases which have been calculated, e.g. $\varepsilon \geqslant 0{\cdot}5$, and $T_1 > T_2$. In other words, if we are interested only in the rate of heat transfer, the Rosseland diffusion approximation may be applied, as has been done by many aerodynamists, such as Pai [15] and Zel'dovich and Raizer [16]. As clearly shown in Figs. 2 and 4 the radiant heat flux attains its steady-state value almost instantaneously. Thus, we may venture to calculate the total heat flux at the inner wall by

$$q(r_1, t) \cong q(r_1, t)_{\text{pure conduction}} + q_r(r_1, \infty)_{\text{pure radiation}}, \tag{26}$$

where

$$q(r_1, t)_{\text{pure conduction}} = 4N\left[\sum_{n=1}^{\infty} \frac{2}{\lambda_n}\frac{[T_2 J_0(\lambda_n r_1) - T_1 J_0(\lambda_n r_2)]\, J_0(\lambda_n r_2)}{J_0^2(\lambda_n r_1) - J_0^2(\lambda_n r_2)} e^{-N\lambda_n^2 t} + \frac{T_1 - T_2}{r_1 \ln(r_1/r_2)}\right] \tag{27}$$

$$q_r(r_1, \infty) = (T_1^4 - T_2^4)\left[\frac{3r_1}{4}\ln\frac{r_2}{r_1} + \frac{r_1}{r_2}\left(\frac{1}{\varepsilon_2} - \frac{1}{2}\right) + \frac{1}{\varepsilon_1} - \frac{1}{2}\right]^{-1}. \tag{28}$$

Equation (28) was obtained from the diffusion approximation [17]. Calculated results from (26) are about 5 per cent higher than those obtained from (16). If, however, we use the formula recommended by Greif and Clapper [2] for the radiant heat flux, i.e.

$$q(r_1, \infty)_{\text{pure radiation}} = (T_1^4 - T_2^4)\left[\frac{3r_1}{4}\ln\frac{r_2}{r_1} + \frac{1}{\varepsilon_1} + \frac{r_1}{r_2} \times \left(\frac{1}{\varepsilon_2} - 1\right)\right]^{-1} \tag{29}$$

the calculated results are in very good agreement with those obtained by the iterative solution, as shown in Fig. 4, for $\varepsilon \geqslant 0{\cdot}5$. However, (26) would fail for $\varepsilon < 0{\cdot}5$.

For steady state, effects of optical depth, τ, surface emissivity, ε, temperature ratio T_2/T_1 and the conduction-radiation parameter N on the radiation potential, the temperature distribution and the heat flux are shown in Figs. 5–7. For small optical depths, such as curves 1 and 2, the drop of radiation potentials across the medium, i.e. $\phi(r_1) - \phi(r_2)$, is essentially independent of N. This holds nearly true also for larger optical depths ($\tau \geqslant 1$) with black surfaces. Since (23) indicates that the radiant heat flux depends only on this potential drop, the total heat flux for a large range of values of N can be calculated by using the potential drop for any value of N, say N_0,

$$q(N) = \frac{4N(T_1 - T_2)}{r \ln(r_2/r_1)} + \frac{\phi(r_1, N_0) - \phi(r_2, N_0)}{3r \ln(r_2/r_1)/4} \tag{30}$$

with other parameters being kept constant. Calculated results from (30), as shown in Fig. 7, agree well with those obtained from the iteration solution for $\varepsilon \geqslant 0{\cdot}5$. As can be seen from Fig. 5, the accuracy of (30) decreases with the decrease of ε. Calculated results from the formula of Greif and Clapper are also shown in Fig. 7 and agree very well with those obtained by the present analysis for $\varepsilon \geqslant 0{\cdot}5$.

Equations (19) and (23) indicate that the temperature field and the heat flux both depend only on the change of radiation potential, i.e. $\Delta\phi$. Figures 5 and 6 show that, for given values of ε, τ and T_2/T_1, $\Delta\phi$ varies little with N, whereas T^4 changes rapidly. Therefore, it can be inferred that $\Delta\phi$ is a weak function of T^4. This gives us a great flexibility to approximate T^4 in (5). In other words, we may linearize $T^4(r)$ by a known function, say $T_a(r)$, which does not differ too much from $T(r)$ so that we can write

$$T^4 \cong 4T_a^3 T - 3T_a^4.$$

The function $T_a(r)$ may be taken as the temperature distribution for pure heat conduction for large values of N.

$$T_a(r) = T_2 + (T_1 - T_2)\frac{\ln(r/r_2)}{\ln(r_1/r_2)}. \tag{31}$$

For $N < 0{\cdot}1$ we may multiply T_2 in (31) by a factor larger than 1, depending on the value of N, as in the case of heat transfer in a plane medium [7]. We may also take T_a equal to that of the Rosseland diffusion approximation, as demonstrated in [10] and [18]. Results of the temperature field and the heat flux thus obtained for the annular medium are in good agreement with those obtained from the iterative solution. Since $\phi(r, t)$ is also a weak function of $T^4(r, t)$, it is conceivable that this simple linearization procedure can apply also for the transient state. However, this calculation has not been made.

CONCLUSIONS

The numerical solution of the integral equations is simpler than that of the differential equations and much simpler than that of the rigorously formulated integro-differential equations. Only the Green's function associated with the radiation potential needed to be constructed. All others can then be readily written down, both for one-dimensional and multi-dimensional problems.

The accuracy of this method of analysis which has been shown for plane media [7] is, now, once more substantiated by the calculated results of heat flux which are in good agreement with those reported in [1] and [2].

The use of radiation potential provides a number of advantages. One of them is to show that the non-linearity of this problem is not as severe as it was once considered. Although only a small amount of time was saved in the numerical solution of the linearized differential, or integral equations for the one-dimensional problem, the linearization procedure will simplify considerably the numerical calculation for more complicated problems. These include multi-dimensional energy transfer in conducting and radiating media, either at rest or in motion and will be reported in the future.

If we are interested only in the rate of heat transfer at the inner wall with $\varepsilon > 0{\cdot}5$, the interaction between radiation and conduction can be neglected for both steady and transient states, provided that the inner wall temperature is higher than the outer. For steady state, this has been shown in [2] and also in [6] for a spherical shell.

The great advantage of the present analysis is that it can be readily applied to problems of combined radiation and conduction in non-grey media and of multi-dimensional heat flow [18].

Before we conclude this study, some remarks on the grey approximation for gases with band absorption–emission seem in order. The grey model is a suitable start in the development of the mathematical method for solving non-grey problems and conclusions obtained from the grey case apply qualitatively to non-grey cases.

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APPENDIX

Under the assumptions stated earlier the transfer equation of radiation is well known as

$$\frac{1}{\kappa}\overline{\Omega}\cdot\nabla I^* = -I^* + n^2 I_b^*, \qquad \text{(A.1)}$$

where n is the refractive index and will be assumed equal to unity, the operator, del, is referred to physical coordinates, $\overline{\Omega}$ denotes the unit vector in the direction of propagation of radiation and other quantities are defined in the Nomenclature of this paper. We now integrate each term in (A.1) over the entire solid angle, 4π, and obtain

$$\frac{1}{\kappa}\nabla\cdot\vec{q}_r^* = -4\pi\bar{I}^* + 4\pi I_b^*, \qquad \text{(A.2)}$$

which is exact. If we multiply (A.1) by $\overline{\Omega}$, integrate over the entire solid angle, 4λ, and take the average value of the intensity for the left-hand side of (A.1), we obtain

$$\vec{q}_r^* = -\frac{4\pi}{3\kappa}\nabla\bar{I}^* \qquad \text{(A.3)}$$

Equation (A.3) is approximate and is known as Eddington's first approximation or Milne–Eddington's approximation. Eliminating $\vec{q}^*$ from (A.2) and (A.3) yields

$$\nabla\cdot\left(\frac{1}{\kappa}\nabla\bar{I}^*\right) - 3\kappa I^* = -3\kappa I_b^*. \qquad \text{(A.4)}$$

To formulate the boundary conditions on $\bar{I}^*$, we follow Eddington by writing the average intensity in terms of two parts, one in the forward direction (denoted by a subscript +) and the other in backward direction (denoted by a subscript −) along the normal to the surface, so that

$$\bar{I}^* = (\bar{I}^*_+ + \bar{I}^*_-)/2. \tag{A.5}$$

An energy balance on the surface gives

$$q^*_r(s) = \pi\bar{I}^*_+(s) - \pi\bar{I}^*_-(s) = \varepsilon_s\pi I^*_{b,s} - \varepsilon_s\pi\bar{I}^*_-. \tag{A.6}$$

Applying (A.3) to the surface gives

$$q^*_r(s) = -\frac{4}{3}\frac{\pi}{\kappa}\left(\frac{\partial\bar{I}^*}{\partial n}\right)_s. \tag{A3'}$$

Eliminating $q^*_r(s)$, $\bar{I}^*_+(s)$ and $\bar{I}^*_-(s)$ from the above four equations we obtain the boundary condition on I^* as

$$\left(\frac{\partial\bar{I}^*}{\partial n}\right)_s = h_s\kappa[\bar{I}^*(s) - I^*_{b,s}]. \tag{A.7}$$

where

$$h_s = \frac{3\varepsilon_s}{2(2-\varepsilon_s)}. \tag{A.8}$$

The energy equation for combined radiation and conduction can then be written as

$$\rho c_p\frac{\partial T^*}{\partial t^*} - \nabla\cdot(k\nabla T^*) = \frac{4}{3}\pi\nabla\cdot\left(\frac{1}{\kappa}\nabla\bar{I}^*\right). \tag{A.9}$$

For constant k and κ equations (A.4) and (A.9) become

$$\rho c_p\frac{\partial T^*}{\partial t^*} - k\nabla^2T^* = \frac{4\pi}{3\kappa}\nabla^2\bar{I}^* \tag{A.10}$$

$$\nabla^2\bar{I}^* - 3\kappa^2\bar{I}^* = -\kappa^2 I^*_b. \tag{A.11}$$

Equation (A.10) can also be written as

$$\rho c_p\frac{\partial T^*}{\partial t^*} - k\nabla^2T^* = 4\pi\kappa(\bar{I}^* - I^*_b). \tag{A.12}$$

The heat flux is given by

$$\vec{q}^* = k\nabla T^* - \frac{4\pi}{3\kappa}\nabla\bar{I}^*. \tag{A.13}$$

TRANSPORT DE CHALEUR PERMANENT ET TRANSITOIRE PAR RAYONNEMENT ET PAR CONDUCTION DANS UN MILIEU LIMITÉ PAR DEUX SURFACES CYLINDRIQUES COAXIALES

Résumé—On analyse le transport de chaleur dans un milieu conducteur émetteur et absorbant limité par deux surfaces cylindriques coaxiales infinies. On considère à la fois le régime transitoire et le régime permanent. En accord avec la simplification quasi-permanente et la première approximation d'Eddington pour le transport par rayonnement, le problème est mis sous la forme de deux équations différentes: une pour le potentiel de rayonnement et l'autre pour la température. Elles sont alors transformées en équations intégrales qui sont résolues par la méthode des approximations successives. Une certaine connaissance de l'effet du rayonnement sur le flux de chaleur est apportée par le calcul du potentiel de rayonnement. Pour des surfaces fortement émissives, on trouve que l'interaction du rayonnement avec la conduction a un effet négligeable sur le flux de chaleur total pour le régime transitoire et le régime permanent. On obtient par suite de cela des formules simples pour le calcul des flux de chaleur.

STATIONÄRER UND INSTATIONÄRER WÄRMEÜBERGANG DURCH STRAHLUNG UND LEITUNG IN EINEM, VON ZWEI HOACHSIALEN ZYLINDERN BEGRENZTEN MEDIUM

Zusammenfassung—Der Wärmeübergang in einem leitenden, emittierenden und absorbierenden, durch unendlich lange koaxiale Zylinder begrenzten Medium wird berechnet. Es wird sowohl der instationäre als auch der Stationäre Zustand betrachtet. Entsprechend einer quasistationären Vereinfachung und mit Hilfe der ersten Näherung von Eddington wird das Problem durch zwei Differentialgleichungen beschrieben: eine für das Strahlungspotential und eine für die Temperatur. Diese Gleichungen werden in Integralgleichungen transformiert und iterativ gelöst. Die Berechnung des Strahlungspotentials lässt den Einfluss der Strahlung auf den Wärmestrom erkennen. Im Falle stark emittierender Oberflächen hat die Wechselwirkung zwischen Strahlung und Leitung nur einen vernachlässigbaren Einfluss auf den Gesamtwärmestrom, sowohl im stationären, wie auch im instationären Zustand. Deshalb erhält man einfache Gleichungen für die Berechnung der Wärmeströme.

СТАЦИОНАРНЫЙ И НЕСТАЦИОНАРНЫЙ ТЕПЛООБМЕН ИЗЛУЧЕНИЕМ И ТЕПЛОПРОВОДНОСТЬЮ В СРЕДЕ, ЗАКЛЮЧЕННОЙ МЕЖДУ ДВУМЯ КОАКСИАЛЬНЫМИ ЦИЛИНДРИЧЕСКИМИ ПОВЕРХНОСТЯМИ

Аннотация—Анализируется перенос тепла в теплопроводной, излучающей и поглощающей среде, ограниченнрй двумя коаксиадьными цилиндрическими поверхностями. Рассматриваются как переходные, так и стационарные состояния. В соответствии с квазистационарным упрощением и первой аппроксимацией эддингтона для лучистого переноса задача формулируется двумя различными уравнениями: одно-для потенциала излучения и другое—для температуры. Эти уравнения затем преобразовываются в интегральные уравнения, решаемые методом последовательных приближений. Некоторая особенность влияния излучения на тепловой поток проявляется при расчете потенциала излучения. Найдено, что для сильно излучающих поверхностей взаимодействие излучения с теплопроводностью оказывает пренебрежимо малое влияние на величину полного теплового потока при стационарных и нестационарных состояниях.

Исходя из этого, получены простые формулы для расчета тепловых потоков.